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LETTER TO THE EDITOR

Characters of $\widehat{sl}(4)_k$ fusion algebra at the non-rational level

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Abstract
We construct the fusion ring of a quasi-rational $\widehat{sl}(4)_k$ WZNW theory at generic $k \notin \mathbb{Q}$ level. It is generated by commutative elements in the group ring $\mathbb{Z}[\tilde{W}]$ of the extended affine Weyl group $\tilde{W}$, which extend polynomially the formal characters of finite-dimensional representations of $sl(4)$.

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1. Introduction

The WZNW models at generic (non-rational) level provide examples of quasi-rational conformal field theories. These are theories described by an infinite discrete spectrum of representations of the chiral algebra, here $\mathfrak{g} = \widehat{sl}(n)_k$, and a fusion rule producing a finite number of terms. The study of these quasi-rational fusion rings is motivated by the fact that they determine, upon 'quantization', the fusion rules of the corresponding rational conformal field theories, described by the fractional level admissible representations of $\mathfrak{g}$ [1]. For the latter there is no sensible Verlinde formula at hand, see the two fully worked out $\widehat{sl}(n)_k$ examples so far, $n = 2$ [2–4] and $n = 3$ [5–7][4]. The quasi-rational fusion rings and their characters are also important as part of the data of more general conformal field theories (CFT) with a continuum spectrum, on manifolds with or without boundaries; see, for example, [9] and references therein for the simplest example of generic level $\widehat{sl}(2)_k$ theory.

Consider the 'pre-admissible' set of representations labelled by the highest weights

$$\{\overline{\Lambda} = \bar{y} \cdot (\lambda' - \lambda(k+n)), k \notin \mathbb{Q} |\ \bar{y} \in \overline{W}, \lambda', \lambda \in P_+,\ \text{s.t.}, \langle \lambda, \alpha_i \rangle \delta + \bar{y}(\alpha_i) \in \Delta_+^{\mathrm{re}},\quad i = 1, 2, \ldots, n-1\} \tag{1.1}$$

[4] A different approach to the problem of $\widehat{sl}(2)_k$ fractional level fusion rules, based on novel indecomposable representations, has been recently proposed in [8] and illustrated on the example $k + 2 = 2/3$.

where $\bar{y} \cdot \lambda = \bar{y}(\lambda + \rho) - \rho$ is the shifted action of the Weyl group $\overline{W}$ of the horizontal algebra $\bar{\mathfrak{g}} = sl(n)$, $P_+ = \oplus_i \mathbb{Z}_{\geqslant 0} \overline{\Lambda}_i$ is the chamber of integral dominant weights, $\overline{\Lambda}_i$ denote the fundamental weights of $\bar{\mathfrak{g}}$ and Δ_+^{re} is the set of real positive roots of $\mathfrak{g}$. The 'pre-integrable' subset ($\bar{y} = \mathbf{1}, \lambda = 0, \bar{\Lambda} = \lambda' \in P_+$) has a fusion ring coinciding with the representation ring of the finite-dimensional irreducible representations (irreps) of $\bar{\mathfrak{g}}$. Its structure constants are given by the classical Weyl–Steinberg formula, which can be 'quantized' to recover the fusion rule multiplicities of the integer level integrable representations [10–12]. The main ingredient in both the classical and the 'quantized' versions of this formula is the multiplicity of states of finite-dimensional irreps of $\bar{\mathfrak{g}}$, encoded in their formal characters. However, all these classical data have no direct meaning for the second, labelled by non-integer highest weights of $\bar{\mathfrak{g}}$, subseries of (1.1) ($\lambda' = 0$), we shall deal with in what follows.

Remarkably, in the simplest $\widehat{sl}(2)_k$ case the fusion characters for the subseries $\lambda' = 0$ turn out to be given by the formal characters of finite-dimensional irreps of the super-algebra $osp(2|1)$. Accordingly the quasi-rational fusion ring of the $\widehat{sl}(2)_k$ representations (1.1) coincides with a product of the representation rings of $osp(2|1)$ and $sl(2)$. Their quantized rational counterpart inherits this 'hidden' $\mathbb{Z}_2$—graded structure, first noticed in [4]. The group $\mathbb{Z}_2$ is the Weyl group $\overline{W}$ of $sl(2)$ and the next truly nontrivial case $n = 3$ also exhibits a $\overline{W}$-graded algebraic structure. The specific character formulae established for $n = 3$ do not extend, however, to $n \geqslant 4$, which is why it is important to study the simplest next case $n = 4$ by methods admitting, in principle, a generalization to arbitrary n.

Preliminary results of this work were announced in [13] and to make the paper self-contained in the next section we repeat some of the introductory material. Our main new result is the explicit formula in section 3 for the characters shown to generate a consistent fusion ring.

2. General setting

Consider the subset $\tilde{\mathcal{W}}^{(+)}$ of the extended affine Weyl group $\tilde{W} = \overline{W} \ltimes t_P = W \rtimes A$

$$\tilde{\mathcal{W}}^{(+)} := \{y \in \tilde{W} | y(\alpha_i) \in \Delta_+^{\mathrm{re}} \text{ for } \forall i = 1, \ldots, n-1\}. \tag{2.1}$$

Here t_P is the subgroup of translations in the weight lattice P of $\bar{\mathfrak{g}} = sl(n)$, and A is the cyclic subgroup of $\tilde{W}$ generated by $\gamma = t_{\overline{\Lambda}_1} w_1 w_2 \cdots w_{n-1}$, which keeps invariant the set of simple roots $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_{n-1}\}$ of $\mathfrak{g}$. Accordingly, the set $\tilde{\mathcal{W}}^{(+)}$ is invariant under the left action of A and is represented as the union $\cup_{a \in A} a\mathcal{W}^{(+)}$, where $\mathcal{W}^{(+)} := \tilde{\mathcal{W}}^{(+)} \cap W$.

Let $k \notin \mathbb{Q}$. The subset $\tilde{\mathcal{W}}^{(+)}$ is a fundamental domain in $\tilde{W}$ with respect to the right action of $\overline{W}$ [6]. The set of weights $\Lambda = \tilde{\mathcal{W}}^{(+)} \cdot k\Lambda_0$ (corresponding to the subset $\lambda' = 0$ of (1.1), $y = \bar{y} t_{-\lambda}$) or, equivalently, the subset $\tilde{\mathcal{W}}^{(+)} \subset \tilde{W}$ itself, labels the highest weights of the maximally reducible Verma modules of $\mathfrak{g}$. Indeed for $\Lambda = y \cdot k\Lambda_0$ and $\beta = y(\alpha)$ s.t. $y \in \tilde{\mathcal{W}}^{(+)}$, the Kac–Kazhdan singular vector criterion holds true for any positive root α of $\bar{\mathfrak{g}}$. Here Λ_0 is the fundamental weight of $\mathfrak{g}$ dual to the affine root α_0 and the Kac–Kazhdan reflections are identified with the right action of $\overline{W}$ on $\tilde{W}$, i.e. $w_{y(\alpha)} \cdot \Lambda = y w_\alpha \cdot k\Lambda_0$.

The factorization of the submodules generated by these singular vectors imposes restrictions on the possible three-point couplings which determine the fusion rules in the CFT. In particular, the decoupling of the singular vectors in the Verma modules of highest weights, labelled by the elements $a \in A$, implies that these elements play the role of 'simple currents' of the fusion ring, with fusion consistent with their left action in $\tilde{\mathcal{W}}^{(+)}$, $y \to ay$. However, completely solving the null vector decoupling equations is a difficult problem (see also the Discussion in what follows). Instead we shall recursively construct a fusion ring building on and extending the approach given in [6]. The idea there is to generalize the formal characters of finite-dimensional irreps of $\overline{\mathfrak{g}}$,

$$\overline{\chi}_\lambda = \sum_{\mu\in\Gamma_\lambda} \overline{m}^\lambda_\mu t_{-\mu} \in \mathbb{Z}[t_P], \qquad \overline{\chi}_{w\cdot\lambda} = \det(w)\overline{\chi}_\lambda, \qquad w \in \overline{W}, \lambda \in P_+, \tag{2.2}$$

which are elements in the group ring $\mathbb{Z}[t_P]$. In analogy one can associate with any $y \in \tilde{\mathcal{W}}^{(+)}$ a formal 'character', an element of the group ring $\mathbb{Z}[\tilde{W}]$ of $\tilde{W}$

$$\chi_y = \sum_{z\in\tilde{W}, zy^{-1}\in W} m^y_z\, z, \qquad y \in \tilde{\mathcal{W}}^{(+)}, \tag{2.3}$$

extended to $\tilde{W}$ by

$$\chi_{yw} := \det(w)\chi_y, \qquad y \in \tilde{\mathcal{W}}^{(+)}, w \in \overline{W}. \tag{2.4}$$

In (2.3) m^y_z are integer non-negative multiplicities, yet to be determined. In particular, for $a \in A$ we define $\chi_a = a$, and the simple current fusion

$$\chi_a \chi_x = \chi_{ax} \tag{2.5}$$

implies that $m^{ay}_{az} = m^y_z$. Clearly we can restrict in (2.3) to elements y in $\mathcal{W}^{(+)}$, for which the sum in (2.3) is required to run over $z \in W$, of length not exceeding the length of the 'highest weight' y.

The finite set $\mathcal{G}_y = \{z \in \tilde{W} | m^y_z \neq 0\}$ generalizes the weight diagram Γ_λ in (2.2). With a notion of a generalized weight diagram, consider a formula for the fusion rule multiplicities, generalizing the classical Weyl–Steinberg formula for $\overline{\chi}_\lambda$

$$\chi_x \chi_y = \sum_{z\in\mathcal{G}_x} m^x_z \chi_{zy} = \sum_{z\in\tilde{\mathcal{W}}^{(+)}} N^z_{x,y}\chi_z, \tag{2.6}$$

$$N^z_{x,y} = \sum_{w\in\overline{W}} \det(w) m^x_{zwy^{-1}}. \tag{2.7}$$

The second equality in (2.6) with the multiplicities given in (2.7) is derived as for the usual $sl(n)$ characters, using the symmetry in (2.4) and the fact that $\tilde{\mathcal{W}}^{(+)}$ is a fundamental domain in $\tilde{W}$.

Introduce a map ι of $\tilde{W}$ into the root lattice Q of $\overline{\mathfrak{g}}$ [5,6]

$$\iota : \tilde{W} \ni y = \bar{y}t_{-\lambda} \mapsto n\lambda + \bar{y}^{-1}\cdot 0 \in Q. \tag{2.8}$$

It has the properties

$$\begin{aligned}
&\iota(xy) = \overline{y}^{-1}(\iota(x)) + \iota(y),\\
&\iota(yw) = w^{-1}\cdot\iota(y), \qquad w \in \overline{W},\\
&\iota(ax) = \iota(x), \qquad a \in A,
\end{aligned} \tag{2.9}$$

and the set $\tilde{\mathcal{W}}^{(+)}$ is expressed alternatively as $\tilde{\mathcal{W}}^{(+)} = \{y \in \tilde{W} | \iota(y) \in P_+\}$.

In the $n = 3$ case the coefficients m^y_z in (2.3) are given as

$$m^y_z = \overline{m}^{\iota(y)}_{\iota(z)}, \tag{2.10}$$

$\overline{m}^{\iota(y)}_{\iota(z)}$ being as in (2.2) the multiplicity of the weight $\mu = \iota(z)$ of the representation of $sl(3)$ of highest weight $\lambda = \iota(y)$. Similarly, the fusion coefficients (2.7) are expressed in terms of the structure constants $\overline{N}^{\iota(z)}_{\iota(x)\iota(y)}$ of the $sl(3)$ character ring

$$N^z_{x,y} = \overline{N}^{\iota(z)}_{\iota(x)\iota(y)}. \tag{2.11}$$

The generalized weight diagrams $\mathcal{G}_y$ are determined by (2.10) and thus have the structure of the weight diagrams $\Gamma_{\iota(y)}$ of triality zero $sl(3)$ representations, with multiplicities preserved, but with the weights $\mu \notin \mathrm{Im}(\iota)$ excluded. The same type of formulae hold in the simpler $sl(2)$ case where $|\mathcal{G}_y| = |\Gamma_{\iota(y)}|$.

However, as discussed in [13], the definition of the generalized weight diagram based on (2.10), and hence (2.11), has to be modified in the higher rank cases, since it is not generally consistent with the Weyl–Steinberg formula (2.6). The multiplicities in (2.3) are only restricted by the inequality $m_z^y \leqslant \overline{m}_{\iota(z)}^{\iota(y)}$.

3. Fusion character ring

We denote by $\overline{\mathfrak{W}}$ the character ring of finite-dimensional irreps of $sl(n)$ generated by the formal classical characters $\overline{\chi}_\lambda$ (2.2). They commute with any $w \in \tilde{W}$ because of the invariance of the classical weight multiplicities $\overline{m}_{w(\mu)}^\lambda = \overline{m}_\mu^\lambda$, $w \in \overline{W}$.

Let $x = t_{-\nu}\overline{x} \in \tilde{\mathcal{W}}^{(+)}$. Guided by the $n = 2, 3$ examples we first introduce the following combinations of classical $sl(n)$ characters $\overline{\chi}_\lambda$ times powers of the generator γ of A, parametrized by weights $b \in Q$:

$$\overline{\chi}_x^{(b)} = \det(\overline{x}) \sum_{i=0}^{n-1} \gamma^i \overline{\chi}_{\nu - \overline{\gamma}^i(\overline{\Lambda}_{n-i}+b)+\overline{\gamma}^i\cdot 0}, \tag{3.1}$$

where $\overline{\Lambda}_n := 0$. These elements of the group ring $\overline{\mathfrak{W}}[A]$ are covariant under A:

$$\overline{\chi}_{ax}^{(b)} = a\overline{\chi}_x^{(b)}, \qquad \forall a \in A. \tag{3.2}$$

We recall the expressions for the generalized characters for the $n = 2, 3$ cases for which $\tilde{\mathcal{W}}^{(+)} = At_{-P_+}$ and $\tilde{\mathcal{W}}^{(+)} = At_{-P_+} \cup Aw_0t_{-P_+}$ respectively [6]

$$\begin{aligned} &\chi_x = \overline{\chi}_x^{(0)} = \det(\overline{x})(\overline{\chi}_\nu + \gamma\overline{\chi}_{\nu-\frac{\alpha}{2}}), && n = 2, \\ &\chi_x = \overline{\chi}_x^{(0)} + \chi_{w_0}\overline{\chi}_x^{(\theta)}, \qquad \chi_{w_0} = 2 + w_0 + w_1 + w_2, && n = 3. \end{aligned} \tag{3.3}$$

The square of the A-invariant combination $F := w_0 + w_1 + w_2 = aFa^{-1}$ in (3.3) lies in $\overline{\mathfrak{W}}[A]$.

We now turn to the $sl(4)$ case. The fundamental chamber $\tilde{\mathcal{W}}^{(+)}$ is alternatively represented as $\tilde{\mathcal{W}}^{(+)} \equiv \mathcal{U}t_{-P_+}$, where

$$\mathcal{U} = \{A, Aw_0, Aw_{10}, Aw_{30}, Aw_{310}, Aw_{2310}\} \tag{3.4}$$

is a subset of $\tilde{W}$; its projection $\overline{\mathcal{U}}$ onto the subgroup $\overline{W}$ gives the right cosets of $\bar{A}$. The group A defines an automorphism of W, $w_\alpha \to \gamma w_\alpha \gamma^{-1} = w_{\gamma(\alpha)}$, $\alpha \in \Pi$, with $\gamma(\alpha_j) = \alpha_{j+1}$ for $j = 0, 1, 2, \ldots, n-1$, identifying $\alpha_n \equiv \alpha_0$. Using this we define A-invariants in the group algebra of W, $F_y = F_{aya^{-1}} = aF_ya^{-1}$, $\forall a \in A$,

$$F_{rst\ldots} \equiv F_{w_{rst\ldots}} := \frac{1}{l_{w_{rst\ldots}}} \sum_{a\in A} aw_{rst\ldots}a^{-1}, \qquad w_{rst\ldots} \in W, \tag{3.5}$$

where l_w takes the value 1 or 2 if the sum over A contains four or two different terms, respectively; e.g., $F_0 = w_0+w_1+w_2+w_3$, $F_{13} = w_{13}+w_{02} = F_{20}$. As it is clear from (2.9), the terms in a given F_y have their ι images in an orbit of the cyclic subgroup $\bar{A}$ of $\overline{W}$. In general, $F_xF_y \neq F_yF_x$, but for example, the three elements $Y_0 := F_0$, $Y_{30} := F_{30}+F_{13}$, $Y_{10} := F_{10}+F_{13}$, commute between themselves.

We now introduce a finite set of formal characters χ_y, $y \in \mathcal{W}^{(+)}$, as in (2.3), for all of which we will adopt the definition (2.10). In employing the map (2.8) and comparing with the standard $sl(4)$ weight diagrams one can use the recursive formula for the multiplicity of a weight μ (see, e.g. [14])

$$\overline{m}_\mu = -\sum_{\overline{w}\in\overline{W};\overline{w}\neq\mathbf{1}} \det(\overline{w})\overline{m}_{\mu+\rho-\overline{w}(\rho)}, \tag{3.6}$$

with the weights on the RHS being strictly greater than μ. Using (3.6) we have for $y \in \mathcal{W}^{(+)}$ and of length $l(y) \leqslant 3$

$$\begin{aligned}
\chi_{w_0} &= 3 + F_0 \equiv 3 + Y_0, \qquad \iota(w_0) = (1, 0, 1),\\
\chi_{w_{10}} &= 3 + 2F_0 + F_{13} + F_{10} \equiv 3 + 2Y_0 + Y_{10}, \qquad \iota(w_{10}) = (0, 1, 2),\\
\chi_{w_{30}} &= 3 + 2F_0 + F_{13} + F_{30} \equiv 3 + 2Y_0 + Y_{30}, \qquad \iota(w_{30}) = (2, 1, 0),\\
\chi_{w_{130}} &= 7 + 5F_0 + 4F_{13} + 2F_{30} + 2F_{10} + (F_{121} + F_{130} + F_{213})\\
&=: 7 + 5Y_0 + 2Y_{10} + 2Y_{30} + Y_{130}, \qquad \iota(w_{130}) = (1, 2, 1),\\
\chi_{w_{230}} &= 1 + F_0 + F_{13} + F_{30} + F_{230} \equiv 1 + Y_0 + Y_{30} + \gamma\overline{\chi}_{\bar{\Lambda}_1}, \qquad \iota(w_{230}) = (4, 0, 0),\\
\chi_{w_{210}} &= 1 + F_0 + F_{13} + F_{10} + F_{210} \equiv 1 + Y_0 + Y_{10} + \gamma^3\overline{\chi}_{\bar{\Lambda}_3}, \qquad \iota(w_{210}) = (0, 0, 4),
\end{aligned} \tag{3.7}$$

of dimension 7, 17, 17, 63, 15, 15, respectively (here $(a_1, a_2, a_3) = \sum_i a_i \bar{\Lambda}_i$). Being linear combinations of A-invariant elements they commute with any $a \in A$. To each of these characters we associate a weight diagram which can be identified with a finite subset of the Cayley graph of W (see [13] for a schematic drawing of the latter). In agreement with (2.7) by a direct computation one obtains

$$\begin{aligned}
\chi_{w_0}\chi_{w_0} &= 1 + 2\chi_{w_0} + \chi_{w_{10}} + \chi_{w_{30}}\\
\chi_{w_0}\chi_{w_{10}} &= \chi_{w_0} + 2\chi_{w_{10}} + \chi_{w_{130}} + \chi_{w_{210}}\\
\chi_{w_0}\chi_{w_{30}} &= \chi_{w_0} + 2\chi_{w_{30}} + \chi_{w_{130}} + \chi_{w_{230}},
\end{aligned} \tag{3.8}$$

which serve as algebraic relations restricting the set of characters

$$\mathcal{F} = \{\chi_{w_0}, \chi_{w_{10}}, \chi_{w_{30}}, \chi_{w_{210}}, \chi_{w_{230}}\}. \tag{3.9}$$

Further fusions recover the characters of length 4, in particular the character $\chi_{w_{2130}}$ of dim 177, with $\iota(w_{2130}) = (2, 2, 2)$,

$$\begin{aligned}
\chi_{w_{2130}} &= \chi_{w_{10}}\chi_{w_{30}} - (1 + 2\chi_{w_0} + \chi_{w_{10}} + \chi_{w_{30}} + \chi_{w_{130}})\\
&= 11 + 9F_0 + 8F_{13} + 4F_{10} + 4F_{30} + 3F_{121} + 3F_{130} + 3F_{213} + 2F_{230} + 2F_{210}\\
&\quad + (F_{10} + F_{30} + F_{1213} + F_{1232} + F_{1321} + F_{2321} + F_{0213} + F_{2130})\\
&=: 11 + 9Y_0 + 4Y_1 + 4Y_3 + 3Y_{130} + 2\gamma\overline{\chi}_{\bar{\Lambda}_1} + 2\gamma^3\overline{\chi}_{\bar{\Lambda}_3} + Y_{2130}.
\end{aligned} \tag{3.10}$$

This is the simplest example in which formula (2.10) fails. The expression obtained by fusion corresponds to a weight diagram that is a subset of the one determined by (2.10) and (3.6). It can be summarized by the following rules:

(i) delete all elements longer than the highest weight element;
(ii) decrease the multiplicity of $w_{ijk\ldots}$, determined from (2.10), by the complement to four of the number of different elementary reflections appearing in $w_{ijk\ldots}$.

Example: for w_{2321} the multiplicity (2.10) is decreased by 1 since three of the four reflections appear, while for w_{0213} it is left unchanged; the multiplicity of the identity is decreased by 4.

One obtains, by direct computation, the relations

$$\begin{aligned}
Y_0^2 &= 4 + Y_{10} + Y_{30}\\
Y_{10}^2 &= 2 + 3Y_{30} - Y_{10} + Y_0\gamma^3\overline{\chi}_{\bar{\Lambda}_3} + \gamma^2\overline{\chi}_{\bar{\Lambda}_2}\\
Y_{30}^2 &= 2 + 3Y_{10} - Y_{30} + Y_0\gamma\overline{\chi}_{\bar{\Lambda}_1} + \gamma^2\overline{\chi}_{\bar{\Lambda}_2}\\
Y_0Y_{10} &= 2Y_0 + Y_{130} + \gamma^3\overline{\chi}_{\bar{\Lambda}_3}\\
Y_0Y_{30} &= 2Y_0 + Y_{130} + \gamma\overline{\chi}_{\bar{\Lambda}_1}\\
Y_{10}Y_{30} &= 6 + Y_{2130},
\end{aligned} \tag{3.11}$$

which imply that all five elements $Y_0, Y_{10}, Y_{30}, Y_{130}, Y_{2130}$ commute. Because of the first three equalities the product of any two of these elements is expressed as a linear combination of the same elements plus the identity, with coefficients in the ring $\overline{\mathfrak{W}}[A]$. Alternatively, the relations (3.11) can be rewritten as

$$
\begin{aligned}
&Y_{10} + Y_{30} = Y_0^2 - 4 =: P_2(Y_0) \\
&2Y_{130} = Y_0^3 - 8Y_0 - (\gamma^3 \overline{\chi}_{\bar{\Lambda}_3} + \gamma \overline{\chi}_{\bar{\Lambda}_1}) =: P_3(Y_0) \\
&2Y_{2130} = Y_0^4 - 10Y_0^2 - Y_0(\gamma^3 \overline{\chi}_{\bar{\Lambda}_3} + \gamma \overline{\chi}_{\bar{\Lambda}_1}) + 8 - 2\gamma^2 \overline{\chi}_{\bar{\Lambda}_2} =: P_4(Y_0) \\
&(Y_{10} - Y_{30})(\gamma^3 \overline{\chi}_{\bar{\Lambda}_3} - \gamma \overline{\chi}_{\bar{\Lambda}_1}) = -Y_0^5 + 12Y_0^3 + 2Y_0^2(\gamma^3 \overline{\chi}_{\bar{\Lambda}_3} + \gamma \overline{\chi}_{\bar{\Lambda}_1}) \\
&\qquad + 4Y_0(\gamma^2 \overline{\chi}_{\bar{\Lambda}_2} - 6) =: P_5(Y_0),
\end{aligned}
\tag{3.12}
$$

where $P_k(Y_0)$ are k-order polynomials of Y_0, and furthermore, Y_0 satisfies a sixth-order polynomial equation,

$$
Y_0^6 - 12Y_0^4 - 2cY_0^3 - 4Y_0^2(\gamma^2 \overline{\chi}_{\bar{\Lambda}_2} - 6) + c^2 - 4(1 + \overline{\chi}_\theta) = 0 \tag{3.13}
$$

where $c = \gamma^3 \overline{\chi}_{\bar{\Lambda}_3} + \gamma \overline{\chi}_{\bar{\Lambda}_1}$. The relations (3.11) suggest that the formal characters we look for are given as linear combinations of the six elements $Y_g, g \in \mathcal{U}_\mathbf{1} := \{\mathbf{1}, w_0, w_{10}, w_{30}, w_{130}, w_{2130}\} \subset \mathcal{U}$, with coefficients in the ring $\overline{\mathfrak{W}}[A]$. We define

$$
\begin{aligned}
\chi_x &:= \sum_{g \in \mathcal{U}_\mathbf{1}} \chi_g \sum_b c_{g,b} \overline{\chi}_x^{(b)} = (\overline{\chi}_x^{(0)} - \overline{\chi}_x^{(\theta+\alpha_2)}) + \chi_0(\overline{\chi}_x^{(2\theta)} + \overline{\chi}_x^{(\theta)}) \\
&\quad + \chi_{10} \overline{\chi}_x^{(2\theta-\alpha_1)} + \chi_{30} \overline{\chi}_x^{(2\theta-\alpha_3)} + \chi_{130} \overline{\chi}_x^{(\theta+\alpha_2)} + \chi_{2130} \overline{\chi}_x^{(\theta)}.
\end{aligned}
\tag{3.14}
$$

Choosing $x = g$ for $g \in \mathcal{U}_\mathbf{1}$ (3.14) reproduces the formulae for the characters in (3.7), (3.10). The values of the shifts b are recovered from each of these five basic characters demanding that the first classical character of the quadruplet (3.1) is an identity. This gives $b_g = 0$ for the identity $g = \mathbf{1}$, while $b_g = -\bar{g} \cdot (-\theta)$ for $g = \bar{g} t_{-\theta}$, i.e. $b_g = 2\theta, 2\theta - \alpha_1, 2\theta - \alpha_3, \theta + \alpha_2, \theta$, respectively. In these checks one has to repeatedly use the symmetry (2.2) of the classical characters to cancel abundant terms. The proposed expression (3.14) is justified by the following lemma.

Lemma 1. *The following Pieri-type formulae hold true for the characters defined in (3.14) and any $f \in \mathcal{F}$:*

$$
\chi_f \chi_x = \sum_{w \in \mathcal{G}_f} \chi_{wx}. \tag{3.15}
$$

The first two relations are proved by a direct but tedious computation comparing the products in the LHS with the RHS of (3.15). It is based on the polynomial relations (3.11) satisfied by the invariants Y_g. One also has to use the classical characters multiplication tables of the fundamental characters $\overline{\chi}_{\bar{\Lambda}_i}$, $i = 1, 2, 3$ and $\overline{\chi}_\theta$, which extend to the multiplication rules

$$
\overline{\chi}_\lambda \chi_x = \sum_{\mu \in \Gamma_\lambda} \chi_{t_{-\mu} x}. \tag{3.16}
$$

The third relation is recovered from the second by the symmetry $w_1 \leftrightarrow w_3$. The proof for the last two characters $\chi_{w_{230}}, \chi_{w_{210}}$ uses the fact that they are expressed in terms of the first three characters in (3.9) (cf (3.7)) and the fundamental classical characters $\overline{\chi}_{\bar{\Lambda}_i}$, $i = 1, 2$.

Formulae (3.15) and (3.14) hold for generic x, sufficiently far from the walls of the chamber $\tilde{\mathcal{W}}^{(+)}$, otherwise cancellations occur, as for example, in the examples (3.8). Having the explicit formula (3.14) one can compute the multiplicities in (2.3). Furthermore, we recall that the following proposition was proved in [13] under the assumption that (3.15) holds true:

Proposition 1. *For any $y \in \mathcal{W}^{(+)}$ there is a formal character χ_y obtained recursively, using (3.15), as a polynomial of the commuting 'fundamental' characters in (3.9).*

Using (2.5), which is part of the fundamental multiplication formulae along with (3.15), the proposition is extended to $y \in \tilde{\mathcal{W}}^{(+)}$. Combined with associativity it furthermore allows one to extend (3.15), (2.5) to the multiplication of two arbitrary characters as in the first equality in (2.6), and hence to confirm formula (2.7) for the fusion multiplicities. The proof is a straightforward generalization of the second proof of lemma 4.5 in [6]. What, however, remains to be proved in general is the non-negativity of the multiplicities $N_{x,y}^{z}$ in (2.7); so far we have checked it on numerous examples. Finally, note that as in the $sl(2)$ and $sl(3)$ cases we can assign a finite dimension to any character, sending all $z \to 1$ in (2.3).

4. Discussion

Extending the results of [6] we have found a consistent $\widehat{sl}(4)_k$ fusion ring generated by the formal characters (3.14). To interpret it as the fusion ring of the related quasi-rational WZNW field theory one has to show that the (shifted) generalized weight diagrams of the generating characters in (3.15), (3.7) are consistent with the solution of the equations expressing the decoupling of the corresponding Verma module singular vectors. As in [5, 7], we can use the standard functional realization of the representations of $sl(4)$, in which the generators are represented by differential operators in six variables, see for example, [14]. The resulting systems of partial differential equations are, however, rather involved and we have checked the simplest of them, corresponding to the 'fundamental' representation labelled by $w_{230} = \gamma t_{-\bar{\Lambda}_1}$: the 15 points of the generalized weight diagram in (3.7) are confirmed. We have also partially checked the multiplication rule of the generator χ_{w_0}, choosing a particular target representation for which the system of equations simplifies: once again the generalized weight diagram in (3.7) consisting of four points of multiplicity 1 and one point of multiplicity 3 is confirmed.

In the rational case $k + 4 = 4/p$ (p-odd) the roots of equation (3.13) determine Y_0 (and hence all five generators expressed by the polynomials $P_k(Y_0)$) in terms of the integrable representations fusion ring characters $\overline{\chi}_{\lambda}^{(p)}(\mu)$ at level p. In principle, this should allow the 'quantization' of the general characters in (3.14), as it has been achieved in the $sl(3)$ case in [6].

The method developed here is expected to apply algorithmically to any n starting with the analogues of the set (3.4) and determining the coefficients $c_{g,b}$ in the analogue of (3.14) from the 'fundamental' fusions generalizing (3.15). The sixth-order polynomial will be replaced by a $(n-1)!$-order polynomial. The non-trivial problem that remains is to find a universal formula for the weight multiplicities in (2.6), extending (2.10), which in particular, would allow one to prove the non-negativity of the structure constants in (2.7).

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